Image Enhancement in Frequency Domain

Background

- Any periodic function can be represented as the sum of sines and cosines of different frequencies, multiplied by a different coefficient (Fig 4-1)
  - The sum is called Fourier series
- Functions that are not periodic but whose area under the curve is finite can be represented as the integral of sines and cosines multiplied by a weight function
  - Known as Fourier transform
- Function represented by Fourier transform can be completely reconstructed by an inverse transform with no loss of information
  - Allows working in Fourier domain and return to the original domain without any loss of information

Preliminary concepts

- Complex numbers
  - A complex number C is defined by
    $$C = R + jI$$
  - The conjugate of a complex number C, denoted by C^* is defined by
    $$C^* = R - jI$$
  - May be viewed geometrically as points on a plane (complex plane)
    * Abscissa is the real axis (value of R)
    * Ordinate is the imaginary axis (value of I)
    * C can be represented as point (R, I) in the rectangular coordinate system of complex plane
  - Polar representation of complex numbers is given by
    $$C = |C|(\cos\theta + j\sin\theta)$$
    * $|C| = \sqrt{R^2 + I^2}$ is the magnitude of the vector from the origin to point (R, I)
    * θ is the angle between the vector and the real axis
    $$\tan\theta = I/R \Rightarrow \theta = \tan^{-1}(I/R)$$
  - Using Euler's formula
    $$e^{j\theta} = \cos\theta + j\sin\theta$$
    where $e = 2.71828\ldots$
  - Then, the complex number is given by
    $$C = |C|e^{j\theta}$$
  - The discussion is applicable to complex functions as well
    * The complex function $F(u)$ can be expressed as the sum $F(u) = R(u) + jI(u)$ where $R(u)$ and $I(u)$ are real and imaginary component functions
- Fourier series

– A function $f(t)$ of a continuous variable t that is periodic with period T is expressed as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

- Impulses and their sifting property
  – A unit impulse of a continuous variable t located at $t = 0$, denoted $\delta(t)$, is defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and is constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

  – If t is the time, impulse is viewed as a spike of infinity amplitude and zero duration, with unit area
  – Impulse has the so called sifting property with respect to integration

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

provided that $f(t)$ is continuous at $t = 0$

  – `http://en.wikipedia.org/wiki/Dirac_delta_function`
  – Sifting simply yields the *value* of the function $f(t)$ at the *location* of the impulse
  – Impulse located at an arbitrary point t_0, denoted by $\delta(t - t_0)$ yields the sifting property as

$$\int_{-infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

  – As an example, if $f(t) = \cos(t)$, using the impulse $\delta(t - \pi)$ yields the result $f(\pi) = \cos(\pi) = -1$
  – Let x represent a discrete variable
  – The unit discrete impulse $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

  – It also satisfies the discrete equivalent of identity

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

  – The sifting preoperty for discrete variables has the form

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

  – Using a discrete impulse located at $x = x_0$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

- Figure 4.2
- *Impulse train*
  * Figure 4.3
  * Defined as the sum of infinitely many periodic impulses ΔT units apart

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

- Fourier transform of functions of one continuous variable
  - Given single variable continuous function $f(t)$ of a continuous variable t
  - Fourier transform $F(u)$ is given by

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

    * μ is also a continuous variable
    * t is integrated out, so, $\mathcal{F}\{f(t)\}$ is a function only of μ
    * So, we can write $\mathcal{F}\{f(t)\} = F(\mu)$ or

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

  - The inverse of the Fourier transform is given by

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

    * The above two functions form the Fourier transform pair
  - Fourier transform can be expressed using Euler's formula as

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]dt$$

  - Computing Fourier transform of a function
    * Figure 4.4a
    * Fourier transform is computed as

$$\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt \\
&= \int_{-W/2}^{W/2} Ae^{-j2\pi\mu t}dt \\
&= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} \\
&= \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right] \\
&= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right] \\
&= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}
\end{aligned}$$

      where $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$
    * The result is called the *sinc* function defined as

$$\text{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

- · sinc(0) = 1 and sinc(m) = 0 for all other integer values of m
- · Plot of $F(\mu)$ in Figure 4.4b

– Fourier spectrum or frequency spectrum
  - ∗ For display purpose, work with the magnitude of the transform
  - ∗ Denoted by

$$|F(\mu)| = AT \left| \frac{\sin(\pi\mu W)}{(\pi\mu W)} \right|$$

  - ∗ Figure 4.4c
    - · Plot of $|F(\mu)|$ as a function of frequency
    - · Location of zeroes of both $F(\mu)$ and $|F(\mu)|$ are inversely proportional to the width W of the box function
    - · Height of the lobes decreases as a function of distance from the origin
    - · Function extends to infinity for bothe positive and negative values of μ

– Fourier transform of a unit impulse located at the origin

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t) dt \\ &= e^{-j2\pi\mu 0} \\ &= e^0 \\ &= 1 \end{aligned}$$

  - ∗ Constant in the frequency domain

– Fourier transform of an impulse located at $t = t_0$ is

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt \\ &= e^{-j2\pi\mu t_0} \\ &= \cos(2\pi\mu t_0) - j \sin(2\pi\mu t_0) \end{aligned}$$

  - ∗ Equivalent representation of a unit circle centered on the origin of the complex plane

– Fourier transform of a periodic impulse train
  - ∗ The only difference in the form of forward and inverse Fourier transform is the sign of the exponential
  - ∗ If a function $f(t)$ has the Fourier transform $F(\mu)$, then the latter function evaluated at t, given by $F(t)$ must have the transform $f(-\mu)$
  - ∗ Using this symmetry property and given that the Fourier transform of an impulse $\delta(t - t_0)$ is $e^{-j2\pi\mu t_0}$, the function $e^{-j2\pi t_0 t}$ has the transform $\delta(-\mu - t_0)$
  - ∗ Let $-t_0 = a$
  - ∗ The transform of $e^{j2\pi at}$ is $\delta(-\mu + a)$
    - · δ is not zero only when $\mu = a$
  - ∗ Impulse train $s_{\Delta T}(t)$ is periodic with period ΔT and can be expressed as a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{\Delta T} t}$$

where

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j \frac{2\pi n}{\Delta T} t} dt$$

   * The integral in the interval $[-\Delta T/2, \Delta T/2]$ encompasses only the impulse of $s_{\Delta T}(t)$ located at the origin, giving

$$\begin{aligned} c_n &= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt \\ &= \frac{1}{\Delta T} e^0 \\ &= \frac{1}{\Delta T} \end{aligned}$$

   * Fourier series expansion becomes

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

   * Fourier transform of a sum is same as the sum of transforms of individual components, that are exponents

$$\mathcal{F}\{e^{j\frac{2\pi n}{\Delta T}t}\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

   * The Fourier transform of periodic impulse train is

$$\begin{aligned} S(\mu) &= \mathcal{F}\{s_{\Delta T}(t)\} \\ &= \mathcal{F}\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\ &= \frac{1}{\Delta T}\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right) \end{aligned}$$

   * Fourier transform of an impulse train with period ΔT is also an impulse train with period $1/\Delta T$

- Convolution
  - Convolution of two functions involves flipping one function about the origin and sliding it past the other
  - Now, we are interested in the convolution of two continuous functions $f(t)$ and $h(t)$ of one continuous variable t using integration instead of summation
  - The convolution, denoted by $\star$, is defined as

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

    where the minus sign indicates flipping, t is the displacement to slide one function past the other, and τ is a dummy variable for integration
  - Fourier transform of above convolution is given by

$$\begin{aligned} \mathcal{F}\{f(t) \star h(t)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi\mu t} dt\right] d\tau \end{aligned}$$

   * The term inside brackets is Fourier transform of $h(t-\tau)$
   * Also, Fourier transform of $h(t)$ is denoted by $H(\mu)$ and

$$\mathcal{F}\{h(t-\tau)\} = H(\mu) e^{-j2\pi\mu\tau}$$

$$\begin{aligned}\mathcal{F}\left\{f(t)\star h(t)\right\} &= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu t}\right]d\tau \\ &= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}d\tau \\ &= H(\mu)F(\mu)\end{aligned}$$

- This equation tells us that the Fourier transform of the convolution of two functions in the spatial domain is equal to the product of the Fourier transform of two functions in frequency domain
- Conversely, we can obtain the convolution in the spatial domain by computing the inverse Fourier transform from the product of the transforms
- **Convolution Theorem:** $f(t) \star h(t)$ ***and*** $H(u)F(u)$ ***are a Fourier transform pair*** written as

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

- The second half of the convolution theorem states:

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

ondicating the convolution in the frequency domain is equivalent to multiplication in the spatial domain

Sampling and the Fourier transform of sampled functions

- Sampling
  - Used to convert continuous functions into a sequence of discrete values
  - Figure 4.5
    - Continuous function $f(t)$ to be sampled at uniform intervals ΔT of independent variable t
    - Model sampling by multiplying $f(t)$ by a sampling function equal to an impulse train ΔT units apart
  - Sampling function

$$\begin{aligned}\tilde{f}(t) &= f(t)s_{\Delta T}(t) \\ &= \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)\end{aligned}$$

    - Each component of summation is an impulse weighted by the value of $f(t)$ at the location of impulse
    - The value f_k of an arbitrary sample in the sequence is computed by integration (strength of weighted impulse) at the point

$$\begin{aligned}f_k &= \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt \\ &= f(k\Delta T)\end{aligned}$$

- Fourier transform of sampled functions
  - Let $F(\mu)$ be the Fourier transform of continuous function $f(t)$
  - Correponding sampled function $\tilde{f}(t)$ is the product of $f(t)$ and an impulse train
  - Fourier transform of product of two functions in spatial domain is convolution of the transforms of two functions in frequency domain
  - Fourier transform of sampled function is

$$\begin{aligned}\tilde{F}(\mu) &= \mathcal{F}\{\tilde{f}(t)\} \\ &= \mathcal{F}\{f(t)s_{\Delta T}(t)\} \\ &= F(\mu) \star S(\mu)\end{aligned}$$

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta \left(\mu - \frac{n}{\Delta T} \right)$$

is the Fourier transform of the impulse train $s_{\Delta T}(t)$

- Convolution can be directly obtained as

$$\begin{aligned} \tilde{F}(\mu) &= F(\mu) \star S(\mu) \\ &= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau \\ &= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta \left(\mu - \tau - \frac{n}{\Delta T} \right) d\tau \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta \left(\mu - \tau - \frac{n}{\Delta T} \right) d\tau \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F \left(\mu - \frac{n}{\Delta T} \right) \end{aligned}$$

The final step follows from the sifting property of the impulse

- From above, Fourier transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$ is an infinite, periodic sequence of copies of $F(\mu)$ which is the transform of the original continuous function
  * Separation between copies is given by $1/\Delta T$
  * Even though $\tilde{f}(t)$ is a sampled function, its transform $\tilde{F}(\mu)$ is continuous because of copies of $F(\mu)$ which is continuous
- Figure 4.6
  * Figure 4.6a – Sketch of Fourier transform $F(\mu)$ of function $f(t)$
  * Figure 4.6b – The transform $\tilde{F}(\mu)$ of the sampled function

- Sampling theorem
  - Band-limited function
    * A function $f(t)$ whose Fourier transform is zero for frequencies outside a finite interval $[-\mu_{\max}, \mu_{\max}]$ about the origin
  - Figure 4.7
    * A lower value of $1/\Delta T$ would cause the periods in $\tilde{F}(\mu)$ to merge; a higher value would provide a clean separation between periods
  - We can recover $f(t)$ from its sampled version if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contained in $\tilde{F}(\mu)$, the transform of the sampled function $\tilde{f}(t)$
    * $\tilde{F}(\mu)$ is a continuous and periodic function with period $1/\Delta T$
    * The entire transform can be characterized by one complete period
    * Hence, we can recover $f(t)$ from that single period by using the inverse Fourier transform
  - A single period equal to $F(\mu)$ can be extracted from $\tilde{F}(\mu)$ if the separation between copies is sufficient
    * Sufficient separation is guaranteed if $1/2\Delta T > \mu_{\max}$, or

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

Theorem 1 (Sampling Theorem) *A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.*

  * Conversely, the maximum frequency that can be captured by sampling a signal at a rate $1/\Delta T$ is $\mu_{\max} = 1/2\Delta T$, or Nyquist rate

Fourier transform and frequency domain

- Extension of the transforms to two variables

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

  - Fourier transform of a discrete function of one variable $f(x), x = 0,1,2,\ldots,M-1$ (discrete Fourier transform, or DFT)

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad \text{for } u = 0,1,2,\ldots,M-1$$

  - Inverse DFT

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \qquad \text{for } x = 0,1,2,\ldots,M-1$$

  - By summing for all values of x for each value of u, you get the complete Fourier transform
    * Approximately M^2 summations and multiplications to compute the DFT
    * Like $f(x)$, $F(u)$ is discrete with same number of components as $f(x)$
  - Finite values for digital domain ensure the existence of DFT and its inverse
  - Concept of a frequency domain follows from Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

  - Substituting into the expression for $F(u)$ and noting that $\cos(-\theta) = \cos(\theta)$, we get

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos(2\pi ux/M) - j\sin(2\pi ux/M)]$$

    for $u = 0,1,2,\ldots,M-1$
    * Each term in Fourier transform is composed of the sum of all values of the function $f(x)$
    * Values of $f(x)$ are multiplied by sines and cosines of various frequencies
    * The domain of values of $F(u)$ is called the frequency domain
      · u determines the frequency of the components of the transform
      · x's are summed out and make the same contribution for each value of u
    * Each of the M terms of $F(u)$ is called a frequency component of the transform
    * Frequency component and frequency domain are also referred to as time component and time domain
  - Fourier transform as a "mathematical prism" to separate function into various components
  - Expressing $F(u)$ in polar coordinates

$$F(u) = |F(u)|e^{-j\phi(u)}$$

    * Magnitude $|F(u)| = [R^2(u) + I^2(u)]^{1/2}$
    * Phase angle $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
    * $R(u)$ and $I(u)$ are real and imaginary parts of $F(u)$
  - Power spectrum

    * Square of the Fourier spectrum

$$
\begin{aligned}
P(u) &= |F(u)|^2 \\
&= R^2(u) + I^2(u)
\end{aligned}
$$

    * Also referred to as spectral density

  – Simple 1D example of DFT: Fig 4-2 (Second ed)
    * Discrete $f(x)$ and $F(u)$
    * $M = 1024$, $K = 8$, $A = 1$
    * Spectrum centered at $u = 0$, accomplished by multiplying $f(x)$ by $(-1)^x$ before computing transform
    * Bottom figures with $K = 16$
    * Height of spectrum doubles as area under the curve in spatial domain doubles
    * Number of zeroes in the spectrum in the same interval doubles as the length of the function doubles

- Two-dimensional DFT and its inverse
  – In 2D, DFT of an image $f(x, y)$ of dimensions $M \times N$ is given by

$$
F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}
$$

  – The inverse Fourier transform is given by

$$
f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}
$$

  – Fourier spectrum, phase angle, and power spectrum are defined by

$$
\begin{aligned}
|F(u, v)| &= [R^2(u, v) + I^2(u, v)]^{1/2} \\
\phi(u, v) &= \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right] \\
P(u, v) &= |F(u, v)|^2 \\
&= R^2(u, v) + I^2(u, v)
\end{aligned}
$$

  – Common practice to multiply the input image by $(-1)^{x+y}$ before computing the Fourier transform
  – Due to properties of exponentials, we have the Fourier transform as

$$
\Im[f(x, y)(-1)^{x+y}] = F(u - M/2, v - N/2)
$$

    * The origin of the Fourier transform $F(0, 0)$ is located at $u = M/2$ and $v = N/2$, or center of the area occupied by 2D DFT
    * This area of frequency domain is called frequency rectangle
    * Guaranteed by requiring that M and N are even integers

  – At point $(0, 0)$ we have

$$
F(0, 0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)
$$

  which is the average of the entire image
    * DC component of the spectrum

  – Figure 4-03 (Second ed)
    * White rectangle of 20×40 pixels superimposed on a black background of 512×512 pixels

    * Image multiplied by $(-1)^{x+y}$ before computing Fourier transform to center the spectrum
    * Separation of spectrum zeroes in u-direction is exactly twice the separation of zeroes in v-direction due to 1:2 size ratio of rectangle in the original image

- Filtering in the frequency domain
  - Basic properties of frequency domain
    * *Each* term of $F(u,v)$ contains *all* values of $f(x,y)$ modified by the values of exponential terms
    * Not easy to make direct association between specific components of an image and its transform
    * Frequency is related to rate of change
      · Frequencies in Fourier transform can be associated with patterns of intensity variations in image
      · Slowest varying frequency component $(u = v = 0)$ corresponds to average gray level of image
      · As you move away from origin of transform, low frequencies correspond to slowly varying components of image
      · Farther away from origin, you have higher frequencies corresponding to faster gray level changes
    * Fig 4-04 (Second ed)
      · Fourier spectrum shows prominent components along $\pm 45°$ direction corresponding to sharp edges
  - Basics of filtering in frequency domain
    * Based on following steps
      1. Multiply input image by $(-1)^{x+y}$ to center the transform
      2. Compute DFT of the image $F(u,v)$
      3. Multiply $F(u,v)$ by a filter function $H(u,v)$
      4. Compute inverse DFT of the result
      5. Obtain the real part of the result
      6. Multiply the result by $(-1)^{x+y}$
    * Filter or filter transfer function
      · Suppresses certain frequencies in transform while leaving others unchanged
    * Fourier transform of the output image

$$G(u,v) = H(u,v) \cdot F(u,v)$$

      · Multiplication of H and F is only between corresponding elements
      · $G(0,0) = H(0,0) \cdot F(0,0)$
    * Zero-phase-shift filters
      · Each component of H multiplies both the real and imaginary part of F
      · Filters do not change the phase of the transform
    * Filtered image $= \Im^{-1}[G(u,v)]$
      · Only retain the real part and multiply that by $(-1)^{x+y}$
      · For real input image and real filter function, the imaginary components of inverse transform are all zero
      · In practice, imaginary components may not be zero due to round-off errors and should be ignored
    * Fig 4-05
      · Preprocessing: crop image to its closest even dimensions;
      · Multiply input image by $(-1)^{x+y}$ to center the transform
  - Some basic filters and their properties
    * Changing the average value of an image to zero
      · Make $F(0,0)$ as zero

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

      · Filter is called *notch filter*; constant function with a hole/notch at the origin

        - Fig 4-06
        - Drop in overall gray level
        - Average cannot be zero because negative values cannot be handled; most negative is changed to zero with other values scaled up from that
        - Useful when it is possible to identify spatial image effects caused by specific, localized frequency domain components
      - Low frequencies depict smooth areas while high frequencies show edges and noise
        - Lowpass and highpass filters
        - Fig 4-07
        - Adding a constant to image resulting from high-pass filter; Fig 4-08

- Correspondence between filtering in spatial and frequency domains
  - Convolution theorem
    - Convolution based on the process of moving a mask over pixels to compute a predefined quantity
    - Discrete convolution of two functions $f(x, y)$ and $h(x, y)$ of size $M \times N$ is defined by

$$f(x,y) * h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$

    - Function h is mirrored about the origin
    - Let $F(u, v)$ and $H(u, v)$ denote the Fourier transforms of $f(x, y)$ and $h(x, y)$, respectively
    - $f(x, y) * h(x, y)$ and $F(u, v)H(u, v)$ constitute a Fourier transform pair, or formally

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

    - Analogous result states that convolution in frequency domain reduces to multiplication in the spatial domain, and vice versa

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$

    - The two equivalences constitute the convolution theorem
  - Impulse function of strength A located at (x_0, y_0) is denoted by $A\delta(x - x_0, y - y_0)$ and is defined by

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y)A\delta(x-x_0, y-y_0) = As(x_0, y_0)$$

    - Summation of a function $s(x, y)$ multiplied by an impulse is simply the value of the function at the location of the impulse, multiplied by the strength of the impulse
    - Limits of the summation are the same as the limits spanned by the function
    - $A\delta(x - x_0, y - y_0)$ is also an image of size $M \times N$
      - Image composed of all zeroes except at (x_0, y_0) where the value of image is A
    - Sifting property of the impulse function
      - Let either f or h in the definition of discrete convolution be an impulse function
      - Convolution of a function with an impulse copies the value of that function to the location of the impulse
      - Unit impulse located at the origin is denoted $\delta(x, y)$

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0)$$

  - Tie between filtering in spatial and frequency domains

    * Fourier transform of unit impulse at the origin

$$
\begin{aligned}
F(u,v) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta(x,y)e^{-j2\pi(ux/M+vy/N)} \\
&= \frac{1}{MN}
\end{aligned}
$$

    * Since there is no exponent component, the Fourier transform of an impulse at the origin is a real constant, with phase angle 0
    * Impulse located at a point other than origin has complex component
    * Magnitude of non-origin impulse is the same, with the translation reflected in nonzero phase angle
  - Let $f(x,y) = \delta(x,y)$ and carry out the discrete convolution
    * Again, $\delta(x,y)$ is the unit impulse at the origin
    * We have

$$
\begin{aligned}
f(x,y) * h(x,y) &= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\delta(m,n)h(x-m,y-n) \\
&= \frac{1}{MN}h(x,y)
\end{aligned}
$$

      since the summation variables are m and n
    * Furthermore

$$
\begin{aligned}
f(x,y) * h(x,y) &\Leftrightarrow F(u,v)H(u,v) \\
\delta(x,y) * h(x,y) &\Leftrightarrow \Im[\delta(x,y)]H(u,v) \\
h(x,y) &\Leftrightarrow H(u,v)
\end{aligned}
$$

  - Implications
    * Filters in the spatial and frequency domains constitute a Fourier transform pair
  - All functions discussed above are the same size $M \times N$
    * Filtering in the frequency domain does not help from computational standpoint
    * Smaller filters in spatial domain
    * Filtering more intuitive in frequency domain
    * Fourier transform and its inverse are linear processes
  - Filters based on Gaussian functions
    * Shape of the function is easily specified
    * Both forward and inverse Fourier transforms of a Gaussian function are real Gaussian functions
    * Discussion limited to 1D functions
    * Gaussian filter function in frequency domain $H(u)$

$$H(u) = Ae^{-u^2/2\sigma^2}$$

      where σ is the standard deviation of the Gaussian curve
    * Corresponding filter in spatial domain is

$$h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$

    * Important observations
      1. Both components of Fourier transform pair are Gaussian and real
      2. Functions behave reciprocally with respect to one another

    · When $H(u)$ has a broad profile (large value of σ), $h(x)$ has a narrow profile, and vice versa
    · As σ approaches infinity, $H(u)$ tends towards a constant function and $h(x)$ tends towards an impulse
  * Fig 4-09
    · Positive values of H and h in both frequency and spatial domains for low pass filter (4.09a and 4.09c)
    · Lowpass filtering can be implemented using a mask with all positive coefficients
    · Narrow frequency domain filter attenuates lower frequencies implying a wider filter in the spatial domain
    · Complex filters, or a highpass filter as a difference of Gaussians
    · In frequency domain (4.09b)

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

    where $A \geq B$ and $\sigma_1 > \sigma_2$
    · Corresponding filter in spatial domain (4.09d)

$$h(x) = \sqrt{2\pi}\sigma_1 A^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 B^{-2\pi^2\sigma_2^2x^2}$$

    · Now the spatial filter has both negative and positive values and once the values turn negative, they never turn positive again (moving away from center in 4.09d)
- Computational complexity

Smoothing frequency-domain filters

- Noise and sharp transitions are the high-frequency component
- Remove noise by attenuating the specified range of high frequency components
- Basic model of filtering in frequency domain

$$G(u, v) = H(u, v)F(u, v)$$

- Basic problem to determine the filter $H(u, v)$
- Ideal lowpass filters
  - Cut off all high frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 from the origin
  - Called 2D ideal lowpass filter (ILPF) with transfer function

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{otherwise} \end{cases}$$

    where D_0 is a specified nonnegative quantity, and $D(u, v)$ is the distance from point (u, v) to the center of the frequency rectangle
  - Distance is taken to be Euclidean distance, and for an $M \times N$ image

$$D(u, v) = \sqrt{(u - M/2)^2 + (v - n/2)^2}$$

  - Figure 4-10
    * All frequencies inside the circle of radius D_0 are passed with no attenuation
    * Radially symmetric filter
  - Cutoff frequency
    * Point of transition between $H(u, v) = 0$ abd $H(u, v) = 1$
    * Not realizable through electronic components but ok in computers

        * Nonphysical filter
    - Computing cutoff frequency
        * Compute circles that enclose specified amounts of total image power P_T
        * Obtained by summation of components of power spectrum at each point as

$$P_T = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u, v)$$

            · Power spectrum at a point is given by

$$P(u, v) = R^2(u, v) + I^2(u, v)$$

        * If transform is centered, a circle of radius r with origin at the center of the frequency rectangle encloses α percent of the power

$$\alpha = 100 \left[\sum_u \sum_v P(u, v)/P_T \right]$$

          with summation taken over the points that lie inside the circle
        * Figure 4-11: Circles of radius on power spectrum
        * Figure 4-12: Applying ideal lowpass filters with cutoff frequencies from Figure 4-11
            · Ringing phenomenon in filtered images with smaller radius
        * Ringing and blurring properties of ILPF
            · Using convolution theorem, Fourier transform of original image $(f(x, y)$ and blurred image $g(x, y)$ are related in frequency domain by

$$G(u, v) = H(u, v)F(u, v)$$

            · Corresponding process in spatial domain is given by

$$g(x, y) = h(x, y) * f(x, y)$$

            · Since the cross-section of the ILPF in the frequency domain looks like a box filter, the cross-section of the corresponding spatial filter has the shape of a sinc function
            · Figure 4-13a: ILPF of radius 5; the spatial filter function is shown in 4-13b
            · Filter $h(x, y)$ has a dominant component at the origin and concentric, circular components about the center component
            · Center component is responsible for blurring
            · Concentric components are responsible for ringing characteristics of ideal filters
            · Radius of center component and number of circles per unit distance from the origin are inversely proportional to the value of cutoff frequency of ideal filter
            · Filtered image can have negative values, and may require scaling
            · Figures 4-13c and d show the application of filter on an image with five bright pixels

- Butterworth lowpass filters
    - Transfer function of a Butterworth lowpass filter (BLPF) of order n with cutoff frequency D_0 from the origin is defined as

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$

    - Figure 4-14
    - BLPF does not have a sharp discontinuity like ILPF
    - In the above equation, $H(u, v) = 0.5$ when $D(u, v) = D_0$
    - Figure 4-15: Application of BLPF

- Ringing observed in filters of higher order but not when $n = 1$ or 2
  - Figure 4-16 – comparison between the spatial representations of BLPFs of various orders, using a cutoff frequency of 5 in all cases

- Gaussian lowpass filters
  - Gaussian lowpass filter (GLPF) in two dimensions is given by

$$H(u, v) = e^{-D^2(u,v)/2\sigma^2}$$

    - $D(u, v)$ is the distance from the origin (center of image)
    - σ is a measure of spread of Gaussian curve
  - If $\sigma = D_0$, the filter can be expressed as

$$H(u, v) = e^{-D^2(u,v)/2D_0^2}$$

    where D_0 is the cutoff frequency
  - When $D(u, v) = D_0$, the filter is down to 0.607 of its maximum value
  - Observations
    - Inverse Fourier transform of a Gaussian lowpass filter is also Gaussian
    - A spatial Gaussian filter, obtained by computing the inverse Fourier transform, will have no ringing
    - Figure 4-17
  - Figure 4-18
    - Not as much smoothing as BLPF of order 2 but no ringing either

- Additional examples of lowpass filtering
  - Figure 4-19
    - Optical character recognition
    - Repair broken characters
  - Figure 4-20 – Cosmetic processing
  - Figure 4-21 – Remote sensing
    - Scan lines due to sensor artifacts along the direction of scan

Sharpening frequency domain filters

- Image sharpening achieved by highpass filtering process
  - Attenuates the low frequency components without disturbing the high frequency information in Fourier transform
  - Reverse of lowpass filter

$$H_{\text{hp}}(u, v) = 1 - H_{\text{lp}}(u, v)$$

  - Figure 4-22 – ideal, Butterworth, and Gaussian highpass filters

- Ideal highpass filters
  - A 2D ideal highpass filter (IHPF) is defined as

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{otherwise} \end{cases}$$

    where D_0 is a specified nonnegative quantity, and $D(u, v)$ is the distance from point (u, v) to the center of the frequency rectangle

  - Just like ILPF, IHPF is not physically realizable
  - Ringing problem persists due to the same reason as in ILPF
  - Figure 4-23
  - Figure 4-24
    * Ringing in Figure 4-24a is so severe that it produces distorted and thickened object boundaries

- Butterworth highpass filters
  - Transfer function of a Butterworth highpass filter (BHPF) of order n with cutoff frequency D_0 from the origin is defined as
$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$
  - Figure 4-22 (middle row)
  - Figure 4-25
    * Boundaries are much less distorted compared to IHPF, even for the smallest value of the cutoff frequency

- Gaussian highpass filters
  - Gaussian highpass filter (GHPF) in two dimensions is given by
$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$
  - Figure 4-26

- Laplacian in frequency domain
  - It can be shown that
$$\Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$
  - It follows that
$$\begin{aligned}\Im\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] &= (ju)^2 F(u,v) + (jv)^2 F(u,v) \\ &= -(u^2 + v^2) F(u,v)\end{aligned}$$
    * The expression on the left side is the Laplacian of $f(x,y)$ and we have
$$\Im[\nabla^2 f(x,y)] = -(u^2 + v^2) F(u,v)$$
    * Laplacian can be implemented in frequency domain by the filter
$$H(u,v) = -(u^2 + v^2)$$
  - Assume that the origin of $F(u,v)$ is centered by performing the operation $f(x,y)(-1)^{x+y}$ before taking the transform
  - Center of filter function can be shifted by
$$H(u,v) = -((u - M/2)^2 + (v - N/2)^2)$$
  - Laplacian filtered image in spatial domain is obtained by computing the inverse Fourier transform of $H(u,v)F(u,v)$ as
$$\nabla^2 f(x,y) = \Im^{-1}(-((u - M/2)^2 + (v - N/2)^2) F(u,v))$$
  - Conversely, we can apply convolution theorem to get the Fourier transform pair notation
$$\nabla^2 f(x,y) \Leftrightarrow -((u - M/2)^2 + (v - N/2)^2) F(u,v)$$

  - Interesting properties of spatial domain Laplacian filter by taking the inverse Fourier transform of $H(u, v)$ defined above
    * Figure 4-27
  - Subtracting the Laplacian from the original image gives you the enhanced image

$$g(x, y) = f(x, y) - \nabla^2 f(x, y)$$

  - The filter can be combined into one operation as

$$H(u, v) = 1 + ((u - M/2)^2 + (v - N/2)^2)$$

  - The enhanced image is obtained with a single transform by

$$g(x, y) = \Im^{-1}((1 + ((u - M/2)^2 + (v - N/2)^2))F(u, v))$$

  - Figure 4-28

- Unsharp masking, high-boost filtering and high-frequency emphasis filtering
  - High-boost filtering
    * Increase the contribution of the original image to overall filtered result
    * Generalization of unsharp masking
      · Generate a sharp image by subtracting its blurred version from itself
      · Obtain a highpass-filtered image by subtracting its lowpass filtered version from itself

$$f_{\text{hp}}(x, y) = f(x, y) - f_{\text{lp}}(x, y)$$

    * For high-boost filtering

$$f_{\text{hb}}(x, y) = Af(x, y) - f_{\text{lp}}(x, y)$$

      where $A \geq 1$
    * You can also write it as

$$\begin{aligned} f_{\text{hb}}(x, y) &= (A - 1)f(x, y) + f(x, y) - f_{\text{lp}}(x, y) \\ &= (A - 1)f(x, y) + f_{\text{hp}}(x, y) \end{aligned}$$

    * For $A = 1$, high-boost filtering reduces to highpass filtering
    * For $A >> 1$, image contribution becomes more dominant
    * Also make sure that you normalize the result back by dividing the coefficients by A
  - Fourier domain
    * $F_{\text{hp}}(u, v) = F(u, v) - F_{\text{lp}}(u, v)$
    * $F_{\text{lp}}(u, v) = H_{\text{lp}}(u, v)F(u, v)$
    * The composite filter in frequency domain is

$$H_{\text{hp}}(u, v) = 1 - H_{\text{lp}}(u, v)$$

    * High-boost filtering in frequency domain is

$$H_{\text{hb}}(u, v) = (A - 1) + H_{\text{hp}}(u, v)$$

    * Figure 4-29
  - High-frequency emphasis
    * Multiply highpass filter by a constant and add an offset so that the DC term is not eliminated by the filter
    * Transfer function given by

$$H_{\text{hfe}}(u, v) = a + bH_{\text{hp}}(u, v)$$

      where $a \geq 0$ and $b > a$
    * Reduces to high-boost filtering when $a = (A - 1)$ and $b = 1$
    * When $b > 1$, high frequencies are emphasized
    * Figure 4-30